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Technical Note TN 24

The Lorentz-invariant solutions of the Klein-Gordon equation

by

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§ 1. Introduction

In this report we shall be interested in the Lorentz invariant solutions of the homogeneous and inhomogeneous Klein-Gordon equations

$$(\Box - m^2)\, \varphi(x) = 0, \qquad m > 0$$
$$(\Box - m^2)\, \varphi(x) = -\,\delta(x),$$

where x stands for the Lorentz four-vector x^0, x^1, x^2, x^3, and $\Box$ is the differential operator

$$\Box \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_0^2} .$$

These equations and their invariant solutions play a dominant role in relativistic quantum field theory. The subject has a long history, dating from before 1930, and has been treated by many authors (Pauli, Herzenberg, Dirac, Schwinger, Dyson and many others). In this report I have tried to introduce the invariant solutions in a unified way and to illuminate their physical meaning with as little reliance on a detailed knowledge of relativistic quantum field theory as possible.

The invariant solutions will turn out to be highly singular functions, (tempered) distributions actually, but we shall not venture upon a rigorous treatment à la Schwartz. On the contrary, the treatment will be physical mathematics throughout (to be contrasted with mathematical physics), i.e. we will freely manipulate with expressions like

$$\delta(x) = \frac{1}{(2\pi)^4} \int e^{ikx}\, dk \qquad \text{etc.}$$

However, virtually all the results can be made rigorous in the theory of distributions. [1)]

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1) See e.g. E.M. de Jager Rapport TW 85, 1962.

Various definitions of the invariant solutions are given in the literature which mostly differ by a factor only. However, there are two conventions which may cause confusion.

1. We shall adopt the Lorentz metric

$$x^2 = x.x = x^{o^2} - x^{1^2} - x^{2^2} - x^{3^2} = x^{o^2} - \vec{x}^2 .$$

(Many authors however define $x^2 = \vec{x}^2 - x^{o^2}$).

A Lorentz transformation is a linear transformation of the coordinates which leaves x^2 invariant; we shall consider only so-called proper Lorentz transformations, i.e. we shall not consider reflections.
The lightcone consists of the points for which $x^2=0$. The points outside the lightcone ($x^2 < 0$) are called space like. The points inside the lightcone ($x^2 > 0$) are called time like; the points inside the upper part of the lightcone cannot be transformed into points inside the lower part by a Lorentz transformation (without reflections) and vice versa.

2. Any function $\varphi(x)$ may be splitted into two parts as follows

$$\varphi(x) = \frac{1}{(2\pi)^2}\int \tilde{\varphi}(k)e^{-ikx}d^4k \qquad (kx=k^o x^o - \vec{k}\,\vec{x})$$

$$= \frac{1}{(2\pi)^2}\int_{k^o>0} \tilde{\varphi}(k)e^{-ikx}\,d^4k + \frac{1}{(2\pi)^2}\int_{k^o<0} \tilde{\varphi}(k)e^{-ikx}d^4k$$

$$= \varphi^{(+)}(x) + \varphi^{(-)}(x).$$

(in our applications the value $k^o=0$ causes no trouble).
It is customary to call $\varphi^{(+)}(x)$ resp. $\varphi^{(-)}(x)$ the positive resp. negative frequency part of the function $\varphi(x)$.

A notable exception is the book of Bogoliubov and Shirkov where the opposite assignment is used.

Finally, we shall denote the invariant solutions by the symbol Δ if $m \neq 0$, and by D if $m=0$; Bogoliubov and Shirkov use the opposite notation.

§ 2. Invariant solutions of the homogeneous Klein-Gordon equation

We consider the equation

$$(\Box - m^2)\, \varphi(x) = 0 \qquad (2.1)$$

where $\Box \equiv \frac{\partial^2}{\partial x_1^{\,2}} + \frac{\partial^2}{\partial x_2^{\,2}} + \frac{\partial^2}{\partial x_3^{\,2}} - \frac{\partial^2}{\partial x_0^{\,2}}$.

Introducing the Fourier transforms by

$$\varphi(x) = \frac{1}{(2\pi)^2} \int e^{-ikx}\, \tilde{\varphi}(k) d^4 k, \quad (kx = k^0 x^0 - \vec{k}.\vec{x}) \qquad (2.2)$$

our equation becomes

$$(k^2 - m^2)\, \tilde{\varphi}(k) = 0 \ . \qquad (2.3)$$

To obtain solutions of this equation we remember that the equation $xf(x) = 0$ (one variable) has as its only solutions $f(x) = c\, \delta(x)$ (Dirac). Applying this to equation (2.3) we put

$$\tilde{\varphi}(k) = c(k)\, \delta(k^2 - m^2) \qquad (2.4)$$

where the function $c(k)$ is only important on the two-sheeted hyperboloid $k^2 - m^2 = 0$ where the δ-function does not vanish. The requirement of Lorentz-invariance is:

$\varphi(x) = \varphi(\Lambda x)$, or alternatively, as may be seen from

(2.2) $\tilde{\varphi}(k) = \tilde{\varphi}(\Lambda k)$, where Λ is the Lorentz transformation in x-space. If we confine ourselves to Lorentz transformations which do not change the direction of time, then these transformations do not transform points of one of the sheets of our hyperboloid into points of the other sheet. Therefore, to secure Lorentz-invariance of the solutions (2.4) we must only demand that the function $c(k)$ is a constant on each of the two sheets of the hyperboloid $k^2-m^2=0$, but need not be necessarily the same constant on both. The most general invariant solution is therefore obtained by putting

$$c(k) = c_1\theta(k^o) + c_2\theta(-k^o) \; . \tag{2.5}$$

We conclude that (2.1) has essentially two independent invariant solutions, all other invariant solutions being obtainable from these by linear combination. The simplest are obtained by taking c_1 resp. c_2 equal to zero. With the usual normalization we have

$$\Delta^{(+)}(x) = -\frac{i}{(2\pi)^3}\int \theta(k^o)\, \delta(k^2-m^2)e^{-ikx}\, d^4k \tag{2.6}$$

$$\Delta^{(-)}(x) = \frac{i}{(2\pi)^3}\int \theta(-k^o)\, \delta(k^2-m^2)e^{-ikx}\, d^4k \; . \tag{2.7}$$

By using the formula

$$\delta(k^2-m^2) = \frac{\delta(k^o-\sqrt{\vec{k}^2+m^2}) + \delta(k^o+\sqrt{\vec{k}^2+m^2})}{2\sqrt{(\vec{k}^2+m^2)}} \tag{2.8}$$

we can write this also as follows

$$\Delta^{(+)}(x) = -\frac{i}{2(2\pi)^3}\int e^{i(\vec{k}.\vec{x}-\omega_k t)}\,\frac{d\vec{k}}{\omega_k} \tag{2.9}$$

$$\Delta^{(-)}(x) = \frac{i}{2(2\pi)^3}\, e^{i(\vec{k}.\vec{x}+\omega_k t)}\,\frac{d\vec{k}}{\omega_k} \tag{2.10}$$

where $$\omega_k = \left|\sqrt{\vec{k}^2+m^2}\right| . \tag{2.11}$$

Two other solutions are important; they are obtained by putting $c_1 = -c_2$ resp. $c_1 = c_2$:

$$\Delta(x) = -\frac{i}{(2\pi)^3}\int \varepsilon(k^o)\,\delta(k^2-m^2)e^{-ikx}\,d^4k \qquad (2.12)$$

$$= -\frac{1}{(2\pi)^3}\int e^{i\vec{k}\vec{x}}\sin\,\omega_k t\,\frac{d\vec{k}}{\omega_k}\;,$$

where $\varepsilon(k^o)$ is the so-called sign-function

$$\varepsilon(k^o) = \theta(k^o) - \theta(-k^o)\;, \qquad (2.13)$$

$$\Delta^{(1)}(x) = \frac{1}{(2\pi)^3}\int \delta(k^2-m^2)e^{-ikx}\,d^4k \qquad (2.14)$$

$$= \frac{1}{(2\pi)^3}\int e^{i\vec{k}\vec{x}}\cos\,\omega_k t\,\frac{d\vec{k}}{\omega_k}$$

<u>Properties</u>

One verifies immediately the following properties of the solutions just obtained:

$$\Delta = \Delta^{(+)} + \Delta^{(-)}\;, \quad \Delta^{(1)} = i(\Delta^{(+)} - \Delta^{(-)}).$$

$$\Delta(-x) = -\Delta(x),\quad \Delta^{(1)}(-x) = \Delta^{(1)}(x),\quad \Delta^{(+)}(-x) = -\Delta^{(-)}(x)$$

Δ and $\Delta^{(1)}$ are real, $\Delta^{(+)} = \Delta^{(-)*}$.

Further

$$\left.\begin{aligned} &\Delta(\vec{x},0) = 0 \\ &\frac{\partial\Delta(\vec{x},0)}{\partial x_o} = -\delta(\vec{x}) \\ &\frac{\partial^2\Delta(\vec{x},0)}{\partial x_o^{\,2}} = 0\;. \end{aligned}\right\} \qquad (2.15)$$

We remark that, in view of the fact that any invariant solution is a linear combination of $\Delta^{(+)}$ and $\Delta^{(-)}$, it follows that the solutions Δ and Δ^1 are completely (apart from a factor) determined by the property of being an uneven resp. even function.

§ 3. Invariant solutions of the inhomogeneous Klein-Gordon equation

Now consider the inhomogeneous equation

$$(\Box - m^2)\, \varphi(x) = -\,\delta(x) \tag{3.1}$$

where $\delta(x) = \delta(x_0, x_1, x_2, x_3)$ is the 4-dimensional δ-function of Dirac. Going over to the Fourier transform by (2.2) and using the formula

$$\delta(x) = \frac{1}{(2\pi)^4} \int e^{-ikx}\, d^4k \tag{3.2}$$

we get the equation

$$(k^2 - m^2)\, \tilde{\varphi}(k) = -\,\frac{1}{(2\pi)^2}\,. \tag{3.3}$$

Again, in the case of one variable, a solution of the equation $xf(x) = 1$ is given by $f(x) = P\,\frac{1}{x}$ where P denotes the Cauchy principal value. Hence, in our case, the most general invariant solution of (3.3) will be

$$(-2\pi)^2\, \tilde{\varphi}(k) = P\,\frac{1}{k^2 - m^2} + \left\{ c_1 \theta(k^0) + c_2 \theta(-k^0) \right\} \delta(k^2 - m^2)\,.$$

The various solutions obviously differ by a solution of the homogeneous equation (2.1). For $c_1 = c_2 = 0$ we get the special solution

$$\bar{\Delta}(x) = -\,\frac{1}{(2\pi)^4}\, P \int \frac{1}{k^2 - m^2}\, e^{-ikx}\, d^4k \,. \tag{3.4}$$

Other important special solutions are

$$\Delta_R(x) = -\frac{1}{(2\pi)^4}\int\left\{P\,\frac{1}{k^2-m^2} - i\pi\,\varepsilon(k^o)\,\delta(k^2-m^2)\right\}e^{-ikx}d^4k$$

$$\Delta_A(x) = -\frac{1}{(2\pi)^4}\int\left\{P\,\frac{1}{k^2-m^2} + i\pi\,\varepsilon(k^o)\,\delta(k^2-m^2)\right\}e^{-ikx}d^4k$$

$$\Delta_c(x) = -\frac{1}{(2\pi)^4}\int\left\{P\,\frac{1}{k^2-m^2} - i\pi\,\delta(k^2-m^2)\right\}e^{-ikx}d^4k \qquad (3.5)$$

$$\Delta_{Ac}(x) = -\frac{1}{(2\pi)^4}\int\left\{P\,\frac{1}{k^2-m^2} + i\pi\,\delta(k^2-m^2)\right\}e^{-ikx}d^4k$$

In section 5 we shall enter into the physical meaning of these solutions.

To obtain in an easy manner some important properties of the above solutions as well as their relationship with the invariant solutions of equation (2.1), we shall make use of an integral representation which will be obtained in the next section.

§ 4. Representation of the invariant solutions as complex integrals

We shall follow an heuristic way of approach. The formal solution of equation (3.3) is

$$-(2\pi)^2\,\tilde{\varphi}(k) = \frac{1}{k^2-m^2}$$

which leads in x-language to the integral

$$-\frac{1}{(2\pi)^2}\int\frac{e^{-ikx}}{k^2-m^2}\,d^4k.$$

This integral can be given a meaning by specifying a path of integration in the complex k-plane which avoids the poles of the integrand. In the following we shall always first perform the integration over k^o. Keeping the other three variables real, we have poles in the k^o-plane at

$$k^o = \pm\sqrt{\vec{k}^2 + m^2}. \qquad (4.1)$$

Now, if C is a path in the complex k^o-plane which avoids the poles, the integral

$$\int_C \frac{e^{-ikx}}{k^2-m}\, d^4k$$

makes sense (as a distribution actually), and we may insert it in the Klein-Gordon equation. Now

$$(\Box - m^2)\int_C \frac{e^{-ikx}}{k^2-m^2}\, d^4k = \int_C e^{-ikx}\, d^4k, \qquad (4.2)$$

and the poles have disappeared from the integrand on the right hand side. Therefore, if C is a closed contour, the right hand side vanishes and we have a solution of the homogeneous Klein-Gordon equation. One may verify, by calculating the residues, that the invariant solutions that we obtained in section 2, correspond to the various ways in which the poles can be circumvented as indicated in fig.1.

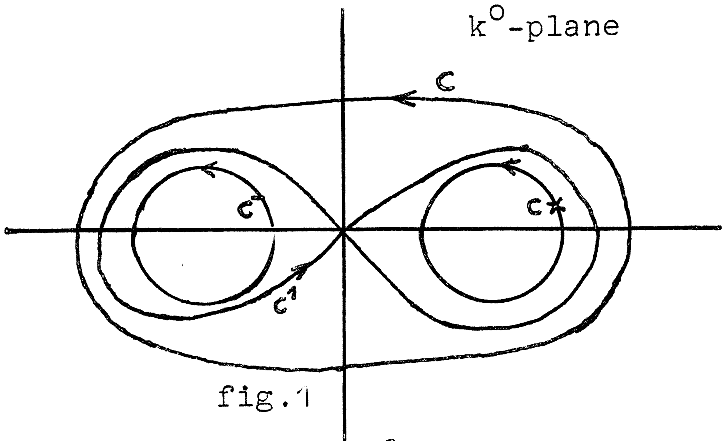


fig.1

$$(\Delta^+, \Delta^-, \Delta, i\Delta^1) = -\frac{1}{(2\pi)^4}\int_{(C^+, C^-, C, C^1)} \frac{e^{-ikx}}{k^2-m^2}\, d^4k. \quad (4.3)$$

However, if we choose a path C which extends to infinity and which can be deformed into the real axis, we get a solution of the inhomogeneous Klein-Gordon equation. Indeed, independent of the way we are avoiding the pole, the right hand side of (4.2) will always be for such a path

$$\int_C e^{-ikx} d^4k = \int_{-\infty}^{\infty} e^{-ikx}\, d^4k = (2\pi)^4\, \delta(x).$$

One may again verify easily that the special solutions (3.4) and (3.5) correspond to the paths shown in fig.2.

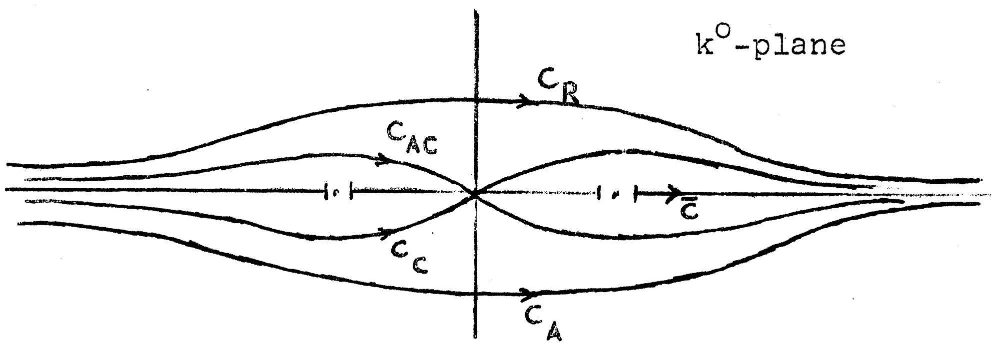


fig.2.

$$(\bar{\Delta}, \Delta_R, \Delta_A, \Delta_C, \Delta_{AC}) = -\frac{1}{(2\pi)^4} \int_{(\bar{C}, C_R, C_A, C_C, C_{AC})} \frac{e^{-ikx}}{k^2-m^2}\, d^4k. \qquad (4.4)$$

For example, the path C_R may be deformed to look like

The contribution of the small semicircles gives rise to the δ-term in (3.5), while the remaining contribution is just the principal value integral.

From the integral representation (4.4) some of the important propert'es of the Δ-functions can be easily seen, if we remark that the exponential

$$e^{-ikx} = e^{-ik^0x^0 + i\vec{k}.\vec{x}}$$

is an exponentially decreasing function in the lower half of the complex k^0-plane if $x^0 > 0$, while it is exponentially decreasing in the lower half plane if $x^0 < 0$. The path of integration in (4.4) may therefore be closed by a big semi-circle in the lower resp. upper half plane. With the help of the representations (4.3) and (4.4) one easily establishes in this way the following relations

$$\Delta_R(x) = \begin{cases} -\Delta(x) & \text{if } x^0 > 0 \\ 0 & \text{if } x^0 < 0 \end{cases} ; \Delta_A(x) = \begin{cases} 0 & \text{if } x^0 > 0 \\ \Delta(x) & \text{if } x^0 < 0 \end{cases} \tag{4.5}$$

$$\Delta_C(x) = \begin{cases} -\Delta^{(+)}(x) & \text{if } x^0 > 0 \\ \Delta^{(-)}(x) & \text{if } x^0 < 0 \end{cases} ; \Delta_{AC}(x) = \begin{cases} -\Delta^{(-)}(x) & \text{if } x^0 > 0 \\ \Delta^{(+)}(x) & \text{if } x^0 < 0 \end{cases}$$

$$2\bar{\Delta}(x) = \Delta_R(x) + \Delta_A(x) = \Delta_C(x) + \Delta_{AC}(x) \tag{4.6}$$

$$\Delta(x) = \Delta_A(x) - \Delta_R(x) \; ; \; i\Delta^1(x) = \Delta_C(x) - \Delta_{AC}(x). \tag{4.7}$$

The function $\Delta_R(x)$ vanishes for negative times, whereas $\Delta_A(x)$ vanishes for positive times. Because they are Lorentz invariant functions this means that they must vanish everywhere outside the lightcone. Indeed, any point outside the lightcone having $x^0 < 0$, can be transformed into a point outside the lightcone having $x^0 > 0$ by a Lorentz transformation, and vice versa. The same is <u>not</u> true for the points inside the lightcone (if we exclude reflections).

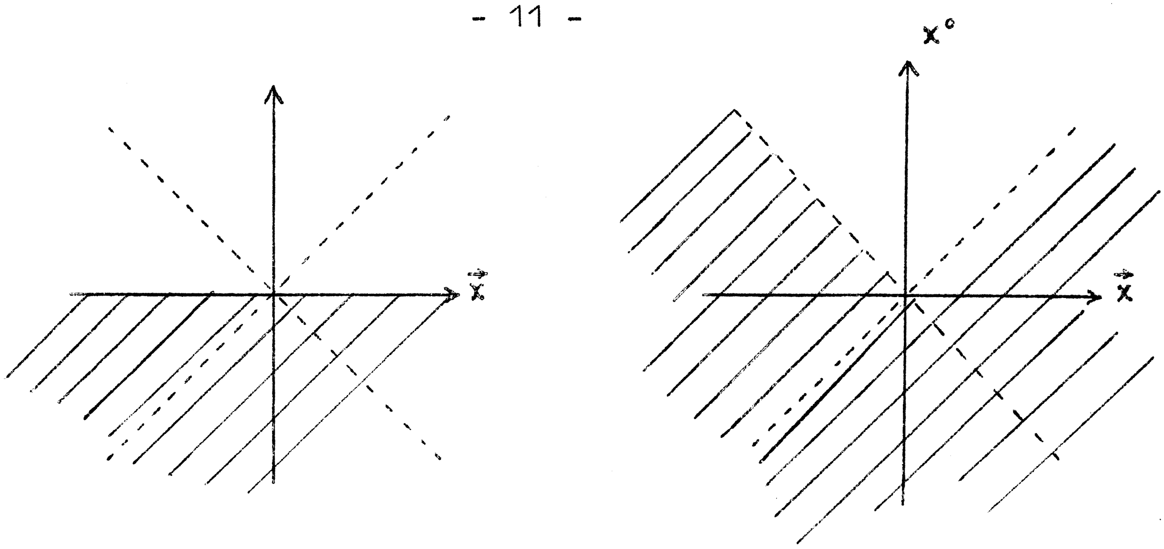


fig.3

The vanishing of an invariant function for $x^0 < 0$ implies its vanishing in a greater region.

Hence, the functions Δ_R and Δ_A vanish everywhere outside the lightcone and it follows from (4.6) and (4.7) that the same is true for the functions $\Delta(x)$ and $\tilde{\Delta}(x)$.

Many other relations between the invariant functions may be written down by using the integral representations. We only mention the following ones (see fig.2):

$$(\Delta_C)^{(+)} = (\Delta_R)^{(+)} \quad ; \quad (\Delta_C)^{(-)} = (\Delta_A)^{(-)} \qquad (4.8)$$

In conclusion of this section we remark that the integrals (4.4) can be represented in the following way, which we shall discuss in section 6.

$$\Delta_R(x) = -\frac{1}{(2\pi)^4} \lim_{\varepsilon\to+0} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{(k^0+i\varepsilon)^2-\vec{k}^2-m^2}\, d^4k$$

$$\Delta_A(x) = -\frac{1}{(2\pi)^4} \lim_{\varepsilon\to+0} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{(k^0-i\varepsilon)^2-\vec{k}^2-m^2}\, d^4k$$

$$\Delta_C(x) = -\frac{1}{(2\pi)^4} \lim_{\varepsilon\to+0} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{k_0^2-\vec{k}^2-m^2+i\varepsilon}\, d^4k \qquad (4.9)$$

$$\Delta_{AC}(x) = -\frac{1}{(2\pi)^4} \lim_{\varepsilon\to+0} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{k_0^2-\vec{k}^2-m^2-i\varepsilon}\, d^4k.$$

§ 5. On the physical meaning of the Δ-functions

a) A solution of the homogeneous Klein-Gordon equation

$$(\Box - m^2)\,\varphi(x) = 0 \tag{5.1}$$

is determined by the value of the function itself and of its time derivative at some initial time. We can choose e.g. the initial values $\varphi(\vec{x},0)$ and $\dot{\varphi}(\vec{x},0)$. The solution $\varphi(x)$ will depend in a linear manner on these initial values. To establish this relation we remark that a complete set of solutions of (5.1) is provided by the plane waves

$$\frac{1}{(2\pi)^{3/2}}\, e^{i\vec{k}.\vec{x} \pm i\omega_k t} \qquad \text{where } \omega_k = +\sqrt{\vec{k}^2+m^2}.$$

Indeed, for t=0 this is a complete set of space functions; for t≠0 their time dependence is determined by (5.1) which leaves only the sign of the frequency undetermined. So we may expand any solution of (5.1) as

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \left\{ a_+(\vec{k})\, e^{i\vec{k}\vec{x} - i\omega_k t} + a_-(\vec{k})\, e^{-i\vec{k}\vec{x} + i\omega_k t} \right\} \frac{dk}{2\omega_k} \tag{5.2}$$

The double set of coefficients $a_+(\vec{k})$ and $a_-(\vec{k})$ is determined by giving the functions $\varphi(\vec{x},0)$ and $\dot{\varphi}(\vec{x},0)$. We may express these coefficients in terms of $\varphi(\vec{x},0)$ and $\dot{\varphi}(\vec{x},0)$ by inverting the Fourier integral and then insert the so obtained expressions again in (5.2); one then obtains

(5.3)
$$\varphi(\vec{x},t) = \int \left\{ \left(\frac{\partial}{\partial t'} \Delta(\vec{x}-\vec{x}',t-t')\, \varphi(\vec{x}',t')\right.\right. \\ \left.\left. - \Delta(\vec{x}-\vec{x}',t-t') \frac{\partial}{\partial t'} \varphi(\vec{x}',t') \right\}_{t'=0} d\vec{x}'. \tag{5.3}$$

where $\Delta(x)$ is given by (2.12).

The right hand side is actually independent of t' so that we may drop the subscript t'=0.

The function $\Delta(x)$ thus appears as the propagator of the solutions of (5.1) from their initial values to the future as well as to the past. The fact that $\Delta(x)$ vanishes outside the lightcone (see § 4) then means that a solution of (5.1) whose initial values are localized in a certain region of space, does not spread with a velocity greater than that of light (c=1 in our units).

The functions $\Delta^{(+)}$ and $\Delta^{(-)}$ play the role of propagators of fields which contain positive resp. negative frequencies only. For example if $\varphi(x)$ contains positive frequencies only and if we write $\Delta = \Delta^{(+)} + \Delta^{(-)}$ in (5.2) then the contribution of $\Delta^{(-)}$ vanishes and the effective propagator which remains is $\Delta^{(+)}$.

b) We now turn to a consideration of the solutions of the inhomogeneous Klein-Gordon equation

$$(\Box - m^2)\, \varphi(x) = -j(x) \tag{5.4}$$

where we assume the "source"-function j(x) to be restricted to a finite region of space-time. Any solution of this equation can be written in the form

$$\varphi(x) = \varphi_{(\alpha)}(x) + \int \Delta_{(\alpha)}(x-x')j(x')d^4x' \tag{5.5}$$

where $\Delta_{(\alpha)}(x)$ is any solution of (3.1) and $\varphi_{(\alpha)}(x)$ is a solution of the homogeneous equation (5.1).

If we choose for $\Delta_{(\alpha)}$ one of the special solutions (3.5), then the function $\varphi_{(\alpha)}(x)$ which appears in (5.5) has a simple physical meaning. For example, if we take $\Delta_R(x)$, then, on account of the property (4.5): $\Delta_R(x) = 0$ if $t < 0$, the integral in (5.5) vanishes for times which are earlier than the time at which the source-function $j(x)$ is switched on. The function $\varphi_{(R)}(x)$ is therefore the field which was present before the source was switched on; it is the so-called "incoming" field. Similarly, $\varphi_{(A)}$ is the field which remains after the source has been switched off, it is the "outgoing" field.

We have noticed already that the separation of the fields into a positive and negative frequency part is an invariant one. It would therefore be possible, without disturbing the invariance, to treat the positive frequency part of the field in a retarded manner and the negative frequency part in an advanced manner, or vice versa. To this possibility conrrespond the propagators Δ_C and Δ_{AC} (see (4.5)). To make this more explicit, let us write

$$\varphi(x) = \varphi^{(+)}(x) + \varphi^{(-)}(x)$$

$$j(x) = j^{(+)}(x) + j^{(-)}(x) .$$

Equation (5.4) can now be written

$$(\Box - m^2)\ \varphi^{(+)}(x) = -j^{(+)}(x)$$

$$(\Box - m^2)\ \varphi^{(-)}(x) = -j^{(-)}(x).$$

If we solve the first of these equations in the form

$$\varphi^{(+)}(x) = \varphi_R{}^{(+)}(x) + \int \Delta_R(x-x')j^{(+)}(x')d^4x'$$

and the second in the form

$$\varphi^{(-)}(x) = \varphi_A{}^{(-)}(x) + \int \Delta_A(x-x')j^{(-)}(x')d^4x',$$

we get for the complete field

$$\varphi(x) = \varphi_R^{(+)}(x) + \varphi_A^{(-)}(x) + \int \Delta_R^{(+)}x-x')j^{(+)}(x')d^4x'$$

$$+ \int \Delta_A^{(-)}(x-x')j^{(-)}(x')d^4x'$$

because in the integral of Δ_R with $j^{(+)}$ the part $\Delta_R^{(-)}$ gives no contribution, and the same holds for $\Delta_A^{(+)}$ in the second integral. We now remember that we had found relation (4.8):

$$(\Delta_C)^{(+)} = (\Delta_R)^{(+)} \ , \ (\Delta_C)^{(-)} = (\Delta_A)^{(-)},$$

so that we obtain

$$\varphi(x) = \varphi_R^{(+)}(x) + \varphi_A^{(-)}(x) + \int \Delta_C(x-x')j(x')d^4x'$$

$$= \varphi_C(x) + \int \Delta_C(x-x')j(x')d^4x'.$$

The foregoing shows that $\Delta_C(x-x')$ propagates the positive frequency part of the field in a retarded manner while the negative frequency part is propagated in an advanced manner correspondingly, the function $\varphi_C(x)$ is the sum of the incoming positive frequency part and the outgoing negative frequency part.

c) <u>Relation with relativistic quantum theory</u>

In relativistic quantum theory one considers an operator field $\varphi(x)$ (where for every x $\varphi(x)$ represents an operator in Hilbert space), which satisfies the Klein-Gordon equation

$$(\Box - m^2)\ \varphi(x) = 0. \qquad (5.6)$$

If $\varphi(x)$ is a real scalar field we get the theory of free, neutral particles with zero spin and restmass m.

The operator character of the field is specified by postulating the following commutation relation

$$[\varphi(x)\ \varphi(y)]_{-} \equiv \varphi(x)\ \varphi(y) - \varphi(y)\ \varphi(x) = i\,\Delta(x-y). \qquad (5.7)$$

In fact this is, apart from a factor, the only possible invariant "ansatz" if we assume that the commutator is a c-number (i.e. a multiple of the unit operator). In that case it follows from the requirement of translational invariance of the theory that the commutator is a function of (x-y) only. By its definition it is obviously an uneven solution of (5.6), but the only invariant uneven solution of (5.6) is the Δ-function as we have seen at the end of section 2.

By the same sort of argument one finds

$$[\varphi^{(+)}(x),\ \varphi^{(-)}(y)]_{-} = i\,\Delta^{(+)}(x-y) \qquad (5.8)$$

$$[\varphi^{(-)}(x),\ \varphi^{(+)}(y)]_{-} = i\,\Delta^{(-)}(x-y)$$

From (4.5) we see that

$$-\theta(x^{o}-y^{o})\ [\varphi(x),\ \varphi(y)] = i\,\Delta_{R}(x-y)$$

and

$$-\theta(x^{o}-y^{o})\ [\varphi^{(+)}(x),\ \varphi^{(-)}(y)]_{-} + \theta(y^{o}-x^{o})[\varphi^{(-)}(x),\varphi^{(+)}(y)]_{-}$$

$$= i\,\Delta_{C}(x-y) \qquad (5.9)$$

One assumes that there exists a state Φ_{o} in our Hilbert space, the so-called vacuum state, with the property that

$$\varphi^{(+)}(x)\,\Phi_{o} = 0 \quad \text{for all } x; \quad \langle\Phi_{o},\ \Phi_{o}\rangle = 1 \qquad (5.10)$$

This assumption corresponds to the requirement that the energy of our system must be (positive) definite.

Taking the vacuum expectation value of (5.8) one gets on account of (5.10)

$$<[\varphi^{(+)}(x), \varphi^{(-)}(y)]_{-}>_0 = <\varphi^{(+)}(x), \varphi^{(-)}(y)>_0 =$$

$$= <\varphi(x)\ \varphi(y)>_0$$

or

$$<\varphi(x)\ \varphi(y)>_0 = i\,\Delta^{(+)}(x-y)$$
$$<\varphi(y)\ \varphi(x)>_0 = -i\,\Delta^{(-)}(x-y) \qquad (5.11)$$

Also, from (5.9) we get:

$$< -\theta(x^0-y^0)\ \varphi^{(+)}(x)\varphi^{(-)}(y) - \theta(y^0-x^0)\ \varphi^{(+)}(y)\ \varphi^{(-)}(x)>_0 =$$

$$= < i\,\Delta_C(x-y)>_0 = i\,\Delta_C(x-y)$$

which can be written as

$$< \theta(x^0-y^0)\ \varphi(x)\ \varphi(y) + \theta(y^0-x^0)\ \varphi(y)\varphi(x)>_0 = -i\,\Delta_C(x-y) \qquad (5.12)$$

The expression in the brackets is the so-called time-ordered product of the operators $\varphi(x)$ and $\varphi(y)$; we write this as

$$< T\,\varphi(x)\ \varphi(y)>_0 = -i\,\Delta_C(x-y). \qquad (5.13)$$

We have thus shown that the invariant Δ-functions are related to simple expressions in the field operators of relativistic quantum theory.

The condition (5.10) which implies that the energy of the system is positive definite, introduces a distinction between the positive and negative frequency parts of the field $\varphi(x)$. To this corresponds the fact that in quantum field theory it is the propagator $\Delta_C(x)$ which plays the dominant role and not the propagator Δ_R which arises in a natural way in classical field theory.

§ 6. The Δ-functions as boundary values of analytic functions

As the simplest example consider the formulas (4.8). We see that the Fourier transform is in all cases essentially the function

$$\frac{1}{k^2-m^2},$$

which is an analytic function with poles at $k^o=\pm\sqrt{\vec{k}^2+m^2}$. The singular functions are obtained by a certain boundary prescription when k approaches the real axis. For example, if f(x) is a testfunction and $\tilde{f}(k)$ its Fourier transform

$$f(x) = \int \tilde{f}(k) e^{ikx} \, d^4k,$$

then the distribution $\Delta_c(x)$ is defined by

$$\int f(x) \, \Delta_c(x) d^4x = -\lim_{\varepsilon \to +0} \int \frac{\tilde{f}(k)}{k^2-m^2+i\varepsilon} \, d^4k.$$

In x-language things are more complicated; as an example we shall consider the function $\Delta^{(+)}(x)$ in some detail. In section 2 we found

$$\Delta^{(+)}(x) = \frac{-i}{2(2\pi)^3} \int e^{-ikx} \frac{d\vec{k}}{\omega_k},$$

where $k.x= k^o x^o-\vec{k}.\vec{x}$ and $k^o= \omega_k=\left|\sqrt{\vec{k}^2+m^2}\right|$.

We now remark that if η is a real form-vector which lies in the forward lightcone (notation: $\eta \in V_+$) which means that $\eta^o > |\vec{\eta}|$, then the integral

$$\int e^{-ik(x-i\eta)} \frac{d\vec{k}}{\omega_k} \qquad (6.1)$$

is an analytic function of $z=x-i\eta$ for $\eta \in V_+$. This is so because also $k \in V_+$, so that we have for the η-term in the exponential:

$$k.\eta = k^o\eta^o - \vec{k}.\vec{\eta} \geq k^o\eta^o - |\vec{k}||\vec{\eta}| \geq |\vec{k}|(\eta^o - |\vec{\eta}|) =$$
$$= c|\vec{k}|$$

where, because of $\eta \in V_+$, c is a positive number. Therefore, the $\eta \in V_+$ introduces a factor in the integrand of (6.1) which decreases exponentially as $|\vec{k}| \to \infty$, hence the integral and all its derivatives exist.

Again, if f(x) is a test-function, we define the distribution $\Delta^{(+)}(x)$ by

$$\int f(x)\,\Delta^{(+)}(x)d^4x = \lim_{\substack{\eta \to 0 \\ \eta \in V_+}} \int f(x)d^4x \left\{ \frac{-i}{2(2\pi)^3} \int e^{-ik(x-i\eta)} \frac{d\vec{k}}{\omega_k} \right\}. \tag{6.2}$$

The integral (6.1) and its analytic continuation can be calculated explicitly. To do this we start from the special points

$x=0$, $\eta=(\eta_o,0,0,0)$, $\eta_o > 0$ for which (6.1) becomes

$$\int e^{-\sqrt{\vec{k}^2+m^2}\,\eta^o} \frac{d\vec{k}}{\sqrt{\vec{k}^2+m^2}} = 4\pi \int_0^\infty e^{-\sqrt{\vec{k}^2+m^2}\,\eta^o} \frac{k^2dk}{\sqrt{\vec{k}^2+m^2}} .$$

With the substitution $\sqrt{\vec{k}^2+m^2} = m\xi$ we obtain

$$\Delta^{(+)}(\eta_o) = \frac{-im^2}{(2\pi)^2} \int_1^\infty e^{-(m\eta_o)\xi}\sqrt{\xi^2-1}\,d\xi = \frac{-m^2}{8\pi}\frac{H_1^{(1)}(im\eta_o)}{im\eta_o}.$$

Because we know that the left hand side is an analytic function of $x-i\eta$, which moreover, on account of Lorentz invariance, only depends on $\xi = Z^2 = (x-i\eta)^2$, while we know also that the Hankel function is an analytic function of its argument accept at the origin, we conclude to the relation

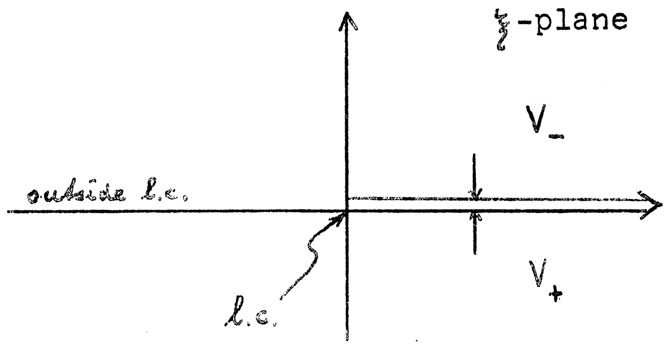
$$\Delta^{(+)}(x-i\eta) = \frac{-m^2}{8\pi} \frac{H_1^{(1)}(m\sqrt{Z^2})}{m\sqrt{Z^2}} \quad \ldots \qquad (6.3)$$

where $\sqrt{Z^2}$ is defined to be positive imaginary at $x=0$, $\eta = (\eta_0, 0, 0, 0)$.

The right hand side of (6.3) is an analytic function of $\xi = Z^2$ in the entire ξ-plane, except for a branchpoint at $\xi=0$ which is due to the square root. Let us investigate the mapping

$$x-i\eta \rightarrow \xi \qquad \eta \in V_+.$$

The space-like points: $x^2 < 0$, $\eta \rightarrow 0$ correspond to the negative real axis of the ξ-plane. The time-like points $x^2 > 0$, $\eta \rightarrow 0$ are mapped on the positive real ξ-axis in such a way that the points inside the upper lightcone: $x^2 > 0$, $x^0 > 0$, $\eta \rightarrow +0$ are obtained when the positive ξ-axis is approached from below and those inside the lower lightcone $x^2 > 0$, $x^0 < 0$, $\eta \rightarrow 0$ when it is approached from above. The points of the lightcone $x^2=0$ itself, are all mapped into the point $\xi=0$.



We have thus defined our distribution $\Delta^{(+)}(x)$ as a boundary value of an analytic function of ξ ; the only points where it has singularities (in the sense of an ordinary function) are the points of the light-cone.

We shall not prove that we actually get a distribution in this way, but for the simple case where m=0 this can easily be seen by using the expansion for the Hankel function. We get

$$D^{(+)}(x) = \lim_{\substack{\eta \to 0 \\ \eta \in V_+}} \frac{i}{(2\pi)^2} \frac{1}{(x-i\eta)^2} . \tag{6.4}$$

On account of Lorentz invariance the four-vector η can be chosen to be $\eta = (\eta_o, 0, 0, 0)$,

$$D^{(+)}(x) = \lim_{\eta_o \to +0} \frac{i}{(2\pi)^2} \frac{1}{(x^o - i\eta^o) - \vec{x}^2}$$

which defines, as we have seen in section 4, the distribution

$$D^{(+)}(x) = \frac{i}{(2\pi)^2} \left\{ P \frac{1}{x^2} + i\pi \, \varepsilon(x^o) \, \delta(x^2) \right\} .$$

We just have seen that $\Delta^{(+)}$ is an analytic function which in general has a branch cut along the positive real axis. The physical meaning of this becomes clear from (5.11) which can be written

$$i\Delta^{(+)}(x) = \langle \varphi(x) \, \varphi(0) \rangle_o .$$

If x is space like, then the operators $\varphi(x)$ and $\varphi(0)$ commute

$$\langle \varphi(x) \, \varphi(0) \rangle_o = \langle \varphi(0) \, \varphi(x) \rangle_o = \langle \varphi(-x) \, \varphi(0) \rangle_o \qquad (x^2 < 0),$$

or $$i\Delta^{(+)}(x) = i\Delta^{(+)}(-x) \qquad \text{for } x^2 < 0,$$

which means that for such points $\Delta^{(+)}(x)$ only depends on x^2 and not on the sign of x^o.
But if x is time like the operators do not commute in general and

$$\Delta^{(+)}(x) \neq \Delta^{(+)}(-x) \qquad \text{for} \quad x^2 > 0$$

i.e. the value of $\Delta^{(+)}(x)$ in the forward lightcone is different from its value in the backward cone.
However, if m=0, we see from (6.4) that in that case no cut exists. This corresponds to the fact that for m=0 the commutator $[\varphi(x)\ \varphi(y)]_-$ vanishes also inside the lightcone and is only different from zero on the lightcone itself.
